

A Resonance Theorem for a Family of Translation Invariant Differentiation Bases

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Abstract

A resonance theorem providing existence of functions that are counterexamples for all members of a given family of translation invariant differentiation bases is proved. Applications of the theorem to Zygmund problem on a choice of coordinate axes are given.

1 Definitions and notation

A mapping B defined on \mathbb{R}^n is said to be a *differentiation basis* if for every $x \in \mathbb{R}^n$, $B(x)$ is a family of bounded measurable sets with positive measure and containing x , such that there exists a sequence $R_k \in B(x)$ ($k \in \mathbb{N}$) with $\lim_{k \rightarrow \infty} \text{diam } R_k = 0$.

For $f \in L(\mathbb{R}^n)$, the numbers

$$\overline{D}_B\left(\int f, x\right) = \overline{\lim}_{\substack{R \in B(x) \\ \text{diam } R \rightarrow 0}} \frac{1}{|R|} \int_R f \quad \text{and} \quad \underline{D}_B\left(\int f, x\right) = \underline{\lim}_{\substack{R \in B(x) \\ \text{diam } R \rightarrow 0}} \frac{1}{|R|} \int_R f$$

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are called *the upper and the lower derivative*, respectively, *of the integral of f at a point x* . If the upper and the lower derivative coincide, then their combined value is called the *derivative of $\int f$ at a point x* and denoted by $D_B(\int f, x)$. We say that the *basis B differentiates $\int f$* (or $\int f$ is differentiable with respect to B) if $\overline{D}_B(\int f, x) = \underline{D}_B(\int f, x) = f(x)$ for almost all $x \in \mathbb{R}^n$. If this is true for each f in the class of functions X we say that B differentiates X .

The *maximal operator M_B* and *truncated maximal operator M_B^r* ($r > 0$) corresponding to a basis B are defined as follows:

$$M_B(f)(x) = \sup_{R \in B(x)} \frac{1}{|R|} \int_R |f|,$$

$$M_B^r(f)(x) = \sup_{\substack{R \in B(x) \\ \text{diam } R < r}} \frac{1}{|R|} \int_R |f|,$$

where $f \in L_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

By \mathbf{I}_n^k ($2 \leq k \leq n$) we will denote the basis such that $\mathbf{I}_n^k(x)$ ($x \in \mathbb{R}^n$) consists of all n -dimensional intervals lengths of which edges take not more than k different values and which contain x . The basis \mathbf{I}_n^n will be denoted by \mathbf{I} . The differentiation with respect to \mathbf{I} is called *strong differentiation*.

A basis B is called:

- *translation invariant* (briefly, TI-basis) if $B(x) = \{x + I : I \in B(0)\}$ for every $x \in \mathbb{R}^n$;
- *homothecy invariant* (briefly, HI-basis) if for every $x \in \mathbb{R}^n, R \in B(x)$ and a homothecy H with the centre at x we have that $H(R) \in B(x)$;
- *formed of sets from the class Δ* if $\overline{B} \subset \Delta$.
- *convex* if it is formed of the class of all convex sets.

Denote by Γ_n the family of all rotations in the space \mathbb{R}^n .

Let B be a basis in \mathbb{R}^n and $\gamma \in \Gamma_n$. The γ -rotated basis B is defined as follows

$$B(\gamma)(x) = \{x + \gamma(I - x) : I \in B(x)\} \quad (x \in \mathbb{R}^n).$$

For an increasing function $\Phi : (0, \infty) \rightarrow (0, \infty)$ and a measurable set $E \subset \mathbb{R}^n$ by $\Phi(L)(E)([\Phi(L)](E))$ we denote the class of all measurable functions

$f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\{f \neq 0\} \subset E$ and $\int_{\{f \neq 0\}} \Phi(|f|) < \infty$ ($\int_{\{f \neq 0\}} \Phi(|f|/h) < \infty$ for some $h \geq 1$).

A function $\Phi : (0, \infty) \rightarrow (0, \infty)$ is said to satisfy Δ_2 -condition at infinity if there are $c > 0$ and $\tau > 0$ such that $\Phi(2t) \leq c\Phi(t)$ for every $t > \tau$.

We say that an increasing function $\Phi : (0, \infty) \rightarrow (0, \infty)$ is *non-regular* if $\overline{\lim}_{t \rightarrow \infty} \frac{\Phi(t)}{t} = \infty$.

The unit cube $(0, 1)^n$ will be denoted by \mathbb{G}^n .

Let $\Phi : (0, \infty) \rightarrow (0, \infty)$ be an increasing function. It is easy to check that:

- 1) If Φ satisfies Δ_2 -condition at infinity, then $[\Phi(L)](\mathbb{G}^n) = \Phi(L)(\mathbb{G}^n)$;
- 2) The class $L \setminus [\Phi(L)](\mathbb{G}^n)$ is non-empty if and only if Φ is non-regular.

The family of all dyadic intervals of order $m \in \mathbb{Z}^n$ we will denote by W_m , i.e.,

$$W_m = \left\{ \times_{j=1}^n \left(\frac{k_j}{2^{m_j}}, \frac{k_j+1}{2^{m_j}} \right) : k_1, \dots, k_n \in \mathbb{Z}^n \right\}.$$

By H_m ($m \in \mathbb{Z}^n$) it will be denoted the family of all possible unions of intervals from W_m .

Below everywhere it will be assumed that the dimension n is greater than 1.

2 Main result

Saks [1] and Busemann and Feller [2] constructed a function $f \in L(\mathbb{R}^n)$ whose integral is not strongly differentiable.

Zygmund [3, p. 99] posed the problem: *Is it possible for arbitrary function $f \in L(\mathbb{G}^2)$ to choose a rotation $\gamma \in \Gamma_2$ so that $\mathbf{I}(\gamma)$ differentiates $\int f$?*

Marstrand [4] gave a negative answer to the problem, namely, constructed a non-negative function $f \in L(\mathbb{G}^2)$ such that for every $\gamma \in \Gamma_2$,

$$\overline{D}_{\mathbf{I}(\gamma)} \left(\int f, x \right) = \infty \text{ almost everywhere on } \mathbb{G}^2.$$

Developing Marstrand approach below we will prove a resonance theorem providing existence of functions that are counterexamples for all members of

a given family of translation invariant bases. Applications of the theorem to Zygmund problem are given also.

The work is a revised version of § II.1 from the monograph [5].

Let Λ be a non-empty family of translation invariant differentiation bases in \mathbb{R}^n and let $\Phi : (0, \infty) \rightarrow (0, \infty)$ be a some function. We will say that Λ has M_Φ -property if for every $h > 1$ and $\varepsilon > 0$ there exist a set $E \subset \mathbb{R}^n$ of positive measure, sets P_B ($B \in \Lambda$) and an interval Q such that:

- 1) $P_B \subset \{M_B^{(\varepsilon)}(h\chi_E) > 1\}$ ($B \in \Lambda$);
- 2) $\{P_B : B \in \Lambda\} \subset H_m$ for some $m \in \mathbb{N}^n$;
- 3) $|P_B| \geq c\Phi(h)|E|$ ($B \in \Lambda$);
- 4) $E \subset Q$ and $P_B \subset Q$ ($B \in \Lambda$);
- 5) $\text{diam } Q < \varepsilon$;
- 6) $|E| \geq c(h)|Q|$,

where $c > 0$, c does not depend on h and ε , $c(h) \in (0, 1)$ and $c(h)$ does not depend on ε .

For a non-empty family of differentiation bases Λ by S_Λ denote the class of all functions $f \in L(\mathbb{G}^n)$ for which

$$\overline{D}_B\left(\int f, x\right) = \infty \text{ almost everywhere on } \mathbb{G}^n$$

for every $B \in \Lambda$.

Theorem 1. *Let Λ be a non-empty family of translation invariant differentiation bases in \mathbb{R}^n and let $\Phi : (0, \infty) \rightarrow (0, \infty)$ be a non-regular increasing function. If Λ has M_Φ -property, then for every $f \in L \setminus [\Phi(L)](\mathbb{G}^n)$ there exists a measure preserving and invertible mapping $\omega : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\{x : \omega(x) \neq x\} \subset \mathbb{G}^n$ and $|f| \circ \omega \in S_\Lambda$. In particular, if Φ additionally satisfies Δ_2 -condition at infinity, then the same conclusion is valid for every $f \in L \setminus \Phi(L)(\mathbb{G}^n)$.*

3 Auxiliary propositions

For $m \in \mathbb{N}^n$ denote

$$W_m^* = \{E \in W_m : E \subset \mathbb{G}^n\}, \quad H_m^* = \{E \in H_m : E \subset \mathbb{G}^n\}.$$

Lemma 1. *Let Λ be a non-empty family of translation invariant differentiation bases in \mathbb{R}^n and $\Phi : (0, \infty) \rightarrow (0, \infty)$ be a some function. If Λ has M_Φ -property, then for every $h > 1$, $\varepsilon > 0$, $m \in \mathbb{N}^n$ and δ with $0 < \delta < c(h)$ there exist a set $E \subset \mathbb{G}^n$ and a family of sets $\{P_B : B \in \Lambda\}$ such that:*

- 1) $\delta/4^n \leq |E| \leq \delta$,
- 2) $P_B \subset \{M_B^{(\varepsilon)}(h\chi_E) > 1\}$ ($B \in \Lambda$),
- 3) $\{P_B : B \in \Lambda\} \subset H_j^*$ for some $j \in \mathbb{N}^n$ with $j \geq m$,
- 4) $|P_B| \geq c\Phi(h)|E|$ ($B \in \Lambda$),
- 5) $|P_B \cap Q| = |P_B||Q|$ ($Q \in W_m^*$, $B \in \Lambda$).

Proof. Let us choose $\eta > 0$ so that

$$\eta < \varepsilon \quad \text{and} \quad 4\left(\frac{1}{\delta c \Phi(h)}\right)^{1/n} < \frac{1}{2^{m_1 + \dots + m_n}}. \quad (1)$$

Due to the definition of M_Φ -property there exist a set $E' \subset \mathbb{R}^n$ with positive measure, sets P'_B ($B \in \Lambda$) and an interval I such that:

$$P'_B \subset \{M_B^{(\varepsilon)}(h\chi_{E'}) > 1\} \quad (B \in \Lambda), \quad (2)$$

$$\{P'_B : B \in \Lambda\} \subset H_j^* \quad \text{for some } j \in \mathbb{N}^n, \quad (3)$$

$$|P'_B| \geq c\Phi(h)|E'| \quad (B \in \Lambda), \quad (4)$$

$$E' \subset I \quad \text{and} \quad P'_B \subset I \quad (B \in \Lambda), \quad (5)$$

$$\text{diam } I < \eta, \quad (6)$$

$$|E'| \geq c(h)|I|. \quad (7)$$

Let \tilde{I} be the interval concentric with I and such that $|E'| = \delta|\tilde{I}|$. (7) implies that $\tilde{I} \supset I$. Put $t = \text{diam } \tilde{I} / \text{diam } I$. Then by virtue of (4) and (5)

$$\frac{|E'|}{\delta} = |\tilde{I}| = t^n |I| \geq t^n c \Phi(h) |E'|.$$

Therefore,

$$t \leq \left(\frac{1}{\delta_C \Phi(h)} \right)^{1/n}. \quad (8)$$

Due to translation invariance of bases $B \in \Lambda$ we can assume that \tilde{I} has the form $(0, a_1) \times \cdots \times (0, a_n)$. By I' denote the smallest among diadic intervals (i.e. among intervals from the family $\bigcup_{i \in \mathbb{Z}^n} W_i$) containing \tilde{I} . Clearly, $I' \subset 4\tilde{I}$.

Therefore

$$\frac{\delta}{4^n} |I'| \leq |E'| \leq \delta |I'| \quad (9)$$

and (see (8), (6) and (1))

$$\text{diam } I' \leq 4 \text{diam } \tilde{I} \leq 4 \left(\frac{1}{\delta_C \Phi(h)} \right)^{1/n} \text{diam } I < \frac{1}{2^{m_1 + \cdots + m_n}}. \quad (10)$$

Let $i \in \mathbb{Z}^n$ be a n -tuple for which $I' \in W_i$. (10) implies that

$$i > m. \quad (11)$$

For each $Q \in W_i^*$, T_Q be the translation mapping I' into Q and put

$$\begin{aligned} E_Q &= T_Q(E') \quad (Q \in W_i^*), \\ P_{B,Q} &= T_Q(P'_B) \quad (B \in \Lambda, \quad Q \in W_i^*), \\ E &= \bigcup_{Q \in W_i^*} E_Q, \\ P_B &= \bigcup_{Q \in W_i^*} P_{B,Q} \quad (B \in \Lambda). \end{aligned}$$

Obviously,

$$E_Q \subset Q \quad \text{and} \quad P_{B,Q} \subset Q \quad (B \in \Lambda, Q \in W_i^*). \quad (12)$$

By virtue of (1), (2) and translation invariance of bases $B \in \Lambda$ we have

$$P_{B,Q} \subset \{M_B^{(n)}(h\chi_{E_Q}) > 1\} \subset \{M_B^{(\varepsilon)}(h\chi_{E_Q}) > 1\} \quad (B \in \Lambda, \quad Q \in W_i^*). \quad (13)$$

Since the intervals from W_i^* are disjoint, then by (3), (4), (9) and (11)–(13) it is easy to conclude that the sets E and P_B ($B \in \Lambda$) satisfy all needed conditions. \square

Lemma 2. *Let f be an increasing function on $[a, b]$ and $\varepsilon > 0$. Then there exist points $h_1 = a < h_2 < \dots < h_k = b$ such that*

$$f(h_{j+1}-) - f(h_j+) \leq \varepsilon \quad (j = 1, \dots, k-1). \quad (14)$$

Proof. Let $h_1 < \dots < h_j$ are chosen. If $h_j = b$, then the construction is completed. If $h_j < b$, then let us take

$$h_{j+1} = \sup \left\{ h \in (h_j, b] : f(h) - f(h_j+) \leq \varepsilon \right\}.$$

After same steps the construction will be completed (in opposite case we will have that $f(b) - f(a) = \infty$). Clearly, the chosen numbers h_1, \dots, h_k satisfy the condition (14). \square

Lemma 3. *Let $\Phi : (0, \infty) \rightarrow (0, \infty)$ be an increasing function, $f \in L(\mathbb{G}^n)$, $k \in \mathbb{N}$, $f/k \notin \Phi(L)(\mathbb{G}^n)$ and $\alpha(h) > 0$ ($h > 1$). Then there exist sets A_j ($j \in \overline{1, m}$) and numbers h_j ($j \in \overline{1, m}$) such that:*

- 1) $A_j \cap A_i = \emptyset$ ($i \neq j$);
- 2) $k < h_j \leq |f(x)|$ ($j \in \overline{1, m}$, $x \in A_j$);
- 3) $0 < |A_j| \leq \alpha(\frac{h_j}{k})$ ($j \in \overline{1, m}$);
- 4) $\sum_{j=1}^m \Phi(\frac{h_j}{k})|A_j| > k$.

Proof. Since $f/k \notin \Phi(L)(\mathbb{G}^n)$, then there are numbers a and b such that

$$k < a < b \quad \text{and} \quad \int_{\{a \leq |f| < b\}} \Phi\left(\frac{|f|}{k}\right) \geq 4k. \quad (15)$$

By virtue of Lemma 2 there are $\lambda_1 = \frac{a}{k} < \lambda_2 < \dots < \lambda_{p+1} = \frac{b}{k}$ with

$$\Phi(\lambda_{q+1}-) - \Phi(\lambda_q+) < 1 \quad (q \in \overline{1, p}). \quad (16)$$

For $q \in \overline{1, p}$ denote

$$E_q = \{|f| = k\lambda_q\} \quad \text{and} \quad E'_q = \{k\lambda_q < |f| < k\lambda_{q+1}\}.$$

Let us choose numbers t_q and τ_q ($q \in \overline{1, p}$) so that

$$\begin{aligned} \lambda_q &< t_q < \tau_q < \lambda_{q+1}, \\ \int_{E'_q \setminus \{kt_q < |f| < k\tau_q\}} \Phi\left(\frac{|f|}{k}\right) &< \frac{1}{p}. \end{aligned} \quad (17)$$

Put $E_q^* = \{kt_q < |f| < k\tau_q\}$ ($q \in \overline{1, p}$). From (16) we have: $\Phi(\tau_q) - \Phi(t_q) < 1$ ($q \in \overline{1, p}$). Therefore for each $q \in \overline{1, p}$ we write

$$\Phi(t_q)|E_q^*| > (\Phi(\tau_q) - 1)|E_q^*| \geq \int_{E_q^*} \Phi\left(\frac{|f|}{k}\right) - |E_q^*|.$$

Consequently (see (15) and (17)),

$$\begin{aligned} \sum_{q=1}^p \Phi(\lambda_q)|E_q| + \sum_{q=1}^p \Phi(t_q)|E_q^*| &\geq \\ &\geq \sum_{q=1}^p \Phi(\lambda_q)|E_q| + \sum_{q=1}^p \left(\int_{E'_q} \Phi\left(\frac{|f|}{k}\right) - \frac{1}{p} - |E_q^*| \right) \geq \\ &\geq \int_{\{a \leq |f| < b\}} \Phi\left(\frac{|f|}{k}\right) - 2 \geq 4k - 2 > k. \end{aligned}$$

Denote

$$N_1 = \{q \in \overline{1, p} : |E_q| > 0\} \text{ and } N_2 = \{q \in \overline{1, p} : |E_q^*| > 0\}.$$

For each $q \in N_1$, $\{E_{q,1}, \dots, E_{q,\nu_q}\}$ be a partition of E_q such that $0 < |E_{q,\nu}| \leq \alpha(\lambda_q)$ ($\nu \in \overline{1, \nu_q}$) and for each $i \in N_2$, $\{E_{i,1}^*, \dots, E_{i,\ell_i}^*\}$ be a partition of E_i^* such that $0 < |E_{i,\ell}^*| \leq \alpha(t_i)$ ($\ell \in \overline{1, \ell_i}$).

Put

$$T = \{E_{q,\nu} : q \in N_1, \nu \in \overline{1, \nu_q}\} \cup \{E_{i,\ell}^* : i \in N_2, \ell \in \overline{1, \ell_i}\}.$$

Let $m = \sum_{q \in N_1} \nu_q + \sum_{q \in N_2} \ell_i$ and $\sigma : \overline{1, m} \rightarrow T$ be a bijection. For $j \in \overline{1, m}$ denote $A_j = \sigma(j)$. Numbers h_j define as follows: $h_j = k\lambda_q$ if $A_j = E_{q,\nu}$ for some q and ν , and $h_j = kt_i$ if $A_j = E_{i,\ell}^*$ for some i and ℓ . It is easy to see that sets A_j and numbers h_j satisfy the needed conditions. \square

Lemma 4. Let $\Phi : (0, \infty) \rightarrow (0, \infty)$ be an increasing function, $f \in L(\mathbb{G}^n)$, $f/h \notin \Phi(L)(\mathbb{G}^n)$ for every $h \geq 1$ and $\alpha(h) > 0$ for every $h > 1$. Then there exist a sequence of measurable sets (A_k) and sequences of positive numbers (h_k) and (q_k) such that:

- 1) $A_k \cap A_m = \emptyset$ ($k \neq m$),
- 2) $q_k < h_k \leq |f(x)|$ ($k \in \mathbb{N}$, $x \in A_k$),
- 3) $\lim_{k \rightarrow \infty} q_k = \infty$,
- 4) $0 < |A_k| \leq \alpha(\frac{h_k}{q_k})$ ($k \in \mathbb{N}$),
- 5) $\sum_{k=1}^{\infty} \Phi(\frac{h_k}{q_k})|A_k| = \infty$.

Proof. It is easy to find sequences of numbers (a_m) and (b_m) such that:

$$0 < a_m < b_m < a_{m+1} \quad (m \in \mathbb{N}), \quad (18)$$

$$\int_{\{a_m < |f| < b_m\}} \Phi\left(\frac{|f|}{m}\right) > m \quad (m \in \mathbb{N}). \quad (19)$$

Let N_i ($i \in \mathbb{N}$) be disjoint infinite subsets of \mathbb{N} and let

$$E_i = \bigcup_{m \in N_i} \{a_m < |f| < b_m\} \quad (i \in \mathbb{N}).$$

From (18) and (19) it follows that the sets E_i are disjoint and $f\chi_{E_i}/h \notin \Phi(L)(\mathbb{G}^n)$ for each $i \in \mathbb{N}$ and $h \geq 1$.

Let i be an arbitrary natural number. Using Lemma 3 for parameters Φ , $f\chi_{E_i}$, i and α we can find sets $A_{i,j} \subset E_i$ ($j \in \overline{1, m_i}$) and numbers $h_{i,j}$ ($j \in \overline{1, m_i}$) with the properties;

- 1) $A_{i,j} \cap A_{i,j'} = \emptyset$ ($j \neq j'$),
- 2) $i < h_{i,j} \leq |f(x)|$ ($j \in \overline{1, m_i}$, $x \in A_{i,j}$),
- 3) $0 < |A_{i,j}| \leq c(\frac{h_{i,j}}{i})$ ($j \in \overline{1, m_i}$),
- 4) $\sum_{j=1}^{m_i} \Phi(\frac{h_{i,j}}{i})|A_{i,j}| > i$.

Making numeration by an index $k \in \mathbb{N}$ of the sequences

$$\begin{aligned} &A_{1,1}, \dots, A_{1,m_1}; A_{2,1}, \dots, A_{2,m_2}; \dots \\ &h_{1,1}, \dots, h_{1,m_1}; h_{2,1}, \dots, h_{2,m_2}; \dots \\ &1, \dots, 1; 2, \dots, 2; \dots \end{aligned}$$

we receive sequences (A_k) , (h_k) and (q_k) that will satisfy all needed conditions. \square

Lemma 5. *Let $j \in \mathbb{N}^n$, $A_1 \in H_j^*$, $A_2 \subset \mathbb{G}^n$ and $|A_2 \cap Q| = |A_2| |Q|$ for each $Q \in W_j^*$. Then $|A_1 \cap A_2| = |A_1| |A_2|$.*

Proof. Let T be the subfamily of W_j^* for which $A_1 = \bigcup_{Q \in T} Q$. Then taking into account the condition of the lemma we have

$$|A_1 \cap A_2| = \sum_{Q \in T} |Q \cap A_2| = \sum_{Q \in T} |Q| |A_2| = |A_1| |A_2|. \quad \square$$

Lemma 6. *Suppose for every $k \in \mathbb{N}$ there are valid conditions: $m_k, j_k \in \mathbb{N}^n$, $m_k \leq j_k \leq m_{k+1}$, $A_k \in H_{j_k}^*$ and $|A_k \cap Q| = |A_k| |Q|$ for each $Q \in W_{m_k}^*$. Then (A_k) is a sequence of independent sets.*

Proof. Let $q \geq 2$ and $k_1 < k_2 < \dots < k_q$. We must prove the equality

$$\left| \bigcap_{\nu=1}^q A_{k_\nu} \right| = \prod_{\nu=1}^q |A_{k_\nu}|. \quad (20)$$

For the case $q = 2$, (20) directly follows from Lemma 5. Let us argue passing from $q - 1$ to q . Assume that (20) is valid for sets $A_{k_1}, \dots, A_{k_{q-1}}$. It is easy to see that

$$\bigcap_{\nu=1}^{q-1} A_{k_\nu} \in H_j^*, \quad \text{where } j = j_{k_{q-1}}.$$

Now taking into account that $m_{k_q} \geq m_{k_{q-1}+1} \geq j_{k_{q-1}}$ and $|A_{k_q} \cap Q| = |A_{k_q}| |Q|$ for each $Q \in W_m^*$, where $m = m_{k_q}$, by virtue of Lemma 5 and induction assumption we write

$$\left| \bigcap_{\nu=1}^q A_{k_\nu} \right| = \left| \bigcap_{\nu=1}^{q-1} A_{k_\nu} \right| |A_{k_q}| = \prod_{\nu=1}^q |A_{k_\nu}|. \quad \square$$

We will need the following well-known result from measure theory (see, e.g., [6, Ch. “Uniform Approximation”] or [7, § 2]).

Theorem A. *For every measurable sets $A_1, A_2 \subset \mathbb{R}^n$ with $|A_1| = |A_2| > 0$ there exists a measure preserving and invertible mapping $\omega : A_1 \rightarrow A_2$.*

4 Proof of Theorem 1

By Lemma 4 there are sequences of sets (A_k) and of positive numbers (h_k) and (q_k) such that:

$$A_k \cap A_m = \emptyset \quad (k \neq m), \quad (21)$$

$$q_k < h_k \leq |f(x)| \quad (k \in \mathbb{N}, \quad x \in A_k), \quad (22)$$

$$\lim_{k \rightarrow \infty} q_k = \infty, \quad (23)$$

$$0 < |A_k| \leq c \left(\frac{h_k}{q_k} \right) \quad (k \in \mathbb{N}),$$

$$\sum_{k=1}^{\infty} \Phi \left(\frac{h_k}{q_k} \right) |A_k| = \infty. \quad (24)$$

According to Lemma 1, for every $k \in \mathbb{N}$ and $m_k \in \mathbb{N}^n$ there exist a set E_k and a family of sets $\{P_{B,k} : B \in \Lambda\}$ with the properties:

$$\{P_{B,k} : B \in \Lambda\} \subset H_{j_k}^* \quad \text{for some } j_k \in \mathbb{N}^n \quad \text{with } j_k \geq m_k, \quad (25)$$

$$\frac{|A_k|}{4^n} \leq |E_k| \leq |A_k|, \quad (26)$$

$$P_{B,k} \subset \left\{ M_B^{(1/k)} \left(\frac{h_k}{q_k} \chi_{E_k} \right) > 1 \right\} = \left\{ M_B^{(1/k)} (h_k \chi_{E_k}) > q_k \right\} \quad (B \in \Lambda), \quad (27)$$

$$|P_{B,k}| \geq c \Phi \left(\frac{h_k}{q_k} \right) |A_k| \quad (B \in \Lambda), \quad (28)$$

$$|P_{B,k} \cap Q| = |P_{B,k}| |Q| \quad (B \in \Lambda, \quad Q \in W_{m_k}^*). \quad (29)$$

From (24), (26) and (28),

$$\sum_{k=1}^{\infty} |P_{B,k}| = \infty \quad (B \in \Lambda). \quad (30)$$

Obviously, we can choose (m_k) so that $m_{k+1} \geq j_k$ ($k \in \mathbb{N}$). Then by (25), (29) and Lemma 6, $(P_{B,k})$ is a sequence of independent sets for every $B \in \Lambda$. Therefore by virtue of (30) and Borel–Cantelli lemma we have

$$\left| \overline{\lim}_{k \rightarrow \infty} P_{B,k} \right| = 1 \text{ for every } B \in \Lambda. \quad (31)$$

Put $g = \sup_{k \in \mathbb{N}} h_k \chi_{E_k}$. Since $g \leq \sum_{k=1}^{\infty} h_k \chi_{E_k}$, then by (21) and (26),

$$\int_{\mathbb{G}^n} g \leq \sum_{k=1}^{\infty} h_k |E_k| \leq \sum_{k=1}^{\infty} h_k |A_k| \leq \int_{\mathbb{G}^n} |f| < \infty.$$

Thus $g \in L(\mathbb{G}^n)$. Let $B \in \Lambda$ and $x \in \overline{\lim}_{k \rightarrow \infty} P_{B,k}$. Then by (23) and (27), $\overline{D}_B(\int g, x) = \infty$. Consequently, taking into account (31) we conclude that $g \in S_{\Lambda}$. Obviously, we have also that

$$g \chi_{\{g < \infty\}} \in S_{\Lambda}. \quad (32)$$

Denote

$$E = \bigcup_{k=1}^{\infty} E_k \text{ and } E'_k = E_k \setminus \bigcup_{j>k} E_j \text{ } (k \in \mathbb{N}).$$

It is easy to check that

$$E'_k \cap E'_m = \emptyset \text{ } (k \neq m),$$

$$g \chi_{\{g < \infty\}} = \sum_{k=1}^{\infty} h_k \chi_{E'_k}.$$

For each $k \in \mathbb{N}$ let us choose a measurable set A'_k so that $A'_k \subset A_k$ and $|A'_k| = |E'_k|$. Denote $A = \bigcup_{k=1}^{\infty} A'_k$. Due to Theorem A there exist a measure preserving and invertible mappings $\omega_k : A'_k \rightarrow E'_k$ ($k \in \mathbb{N}$) and $\omega_0 : \mathbb{G}^n \setminus A \rightarrow \mathbb{G}^n \setminus E$. A mapping ω define as follows

$$\omega(x) = \begin{cases} \omega_k(x) & (k \in \mathbb{N}, x \in A'_k), \\ \omega_0(x) & (x \in \mathbb{G}^n \setminus A), \\ x & (x \in \mathbb{R}^n \setminus \mathbb{G}^n). \end{cases}$$

It is easy to check that: 1) ω is measure preserving and invertible; and 2) $|f| \circ \omega \geq g$. Now taking into account (32) we conclude the validity of the theorem. \square

Remark 1. For the case of finite or countable family Λ , Theorem 1 can be strengthened by achieving divergence at every point with respect to each $B \in \Lambda$, i.e. a mapping ω can be chosen so that for every basis $B \in \Lambda$ the equality $\overline{D}_B(\int |f| \circ \omega, x) = \infty$ would be fulfilled at every $x \in \mathbb{G}^n$.

5 Applications of Theorem 1

5.1. Zygmund problem in general setting may be formulated as follows: *Let B be a translation invariant basis in \mathbb{R}^n and let $\Delta(B) = \{B(\gamma) : \gamma \in \Gamma_n\}$. Is the class $S_{\Delta(B)}$ non-empty?*

Below it is found a quite general condition for a basis B (see Corollary 1) fulfillment of which provide the positive answer to the posed question.

For a basis B denote

$$\Phi_B(h) = \lim_{t \rightarrow \infty} \overline{\lim}_{r \rightarrow 0} \frac{|\{M_B^{(tr)}(h\chi_{V_r}) > 1\}|}{|V_r|} \quad (h > 0),$$

where $V_r = \{x \in \mathbb{R}^n : \text{dist}(x, 0) < r\}$. Note that:

- 1) Φ_B is increasing;
- 2) If B is a convex basis, then by virtue of the estimation $M_B(\chi_{V_r})(x) < cr / \text{dist}(x, V_r)$ ($x \notin V_{2r}$) (see [8, Lemma 1]) we have

$$\Phi_B(h) = \overline{\lim}_{r \rightarrow 0} \frac{|\{M_B^{(tr)}(h\chi_{V_r}) > 1\}|}{|V_r|};$$

- 3) If B is translation and homothecy invariant, then

$$\Phi_B(h) = |\{M_B^{(tr)}(h\chi_V) > 1\}|,$$

where $V = \{x \in \mathbb{R}^n : \text{dist}(x, 0) < 1\}$.

Let us call sets from a class Δ *uniformly measurable in Jordan sense* if for every $\varepsilon > 0$ there exist $k \in \mathbb{N}$ and $\varepsilon > 0$ such that:

- 1) $kr^n < \varepsilon$;
- 2) for every $E \in \Delta$, there exists a cover of ∂E consisting of k balls with radius ε .

Remark 2. If Δ is a collection of measurable in Jordan sense and mutually congruent sets, then it is easy to see that the sets from Δ are uniformly measurable in Jordan sense.

Lemma 7. *Let Δ be a non-empty family of sets that are uniformly measurable in Jordan sense and let $\inf_{E \in \Delta} |E| > 0$. Then for every $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}^n$ such that*

$$\left| \bigcup_{Q \in W_m, Q \subset E} Q \right| > (1 - \varepsilon)|E| \quad \text{and} \quad \left| \bigcup_{Q \in W_m, Q \cap E \neq \emptyset} Q \right| < (1 + \varepsilon)|E|$$

for every $E \in \Delta$ and $m \geq m_0$.

Proof. Denote $t = \inf_{E \in \Delta} |E|$ and $V(x, r) = \{y \in \mathbb{R}^n : \text{dist}(y, x) < r\}$. By virtue of the lemma condition there are $k \in \mathbb{N}$ and $r > 0$ such that

$$4^n k r^n < \varepsilon t, \tag{33}$$

and for every $E \in \Delta$ we can choose points $x_{E,1}, \dots, x_{E,k}$ for which

$$\partial E \subset \bigcup_{j=1}^k V(x_{E,j}, r). \tag{34}$$

Let $m_0 \in \mathbb{N}^n$ be such that $\text{diam } Q < r$ if $Q \in W_{m_0}$. Suppose $m \geq m_0$. For $E \in \Delta$ denote

$$A_E = \left\{ Q \in W_m : Q \cap \bigcup_{j=1}^k V(x_{E,j}, r) \neq \emptyset \right\}.$$

By choosing of m_0 we have

$$\bigcup_{Q \in A_E} Q \subset \bigcup_{j=1}^k V(x_{E,j}, 2r).$$

Consequently, by (33)

$$\left| \bigcup_{Q \in A_E} Q \right| \leq 2^n \sum_{j=1}^k |V(x_{E,j}, r)| \leq 2^n k 2^n r^n < \varepsilon t. \tag{35}$$

Note that if $Q \in W_m$, $Q \cap E \neq \emptyset$ and $Q \cap (\mathbb{R}^n \setminus E) \neq \emptyset$, then $Q \cap \partial E \neq \emptyset$. Therefore, by (34),

$$\{Q \in W_m : Q \subset E\} \supset \{Q \in W_m : Q \cap E \neq \emptyset\} \setminus A_E$$

and

$$\{Q \in W_m : Q \cap E \neq \emptyset\} \subset \{Q \in W_m : Q \subset E\} \cup A_E.$$

Consequently, taking into account (35), we obtain

$$\left| \bigcup_{Q \in W_m, Q \subset E} Q \right| \geq \left| \bigcup_{Q \in W_m, Q \cap E \neq \emptyset} Q \right| - \left| \bigcup_{Q \in A_E} Q \right| > |E| - \varepsilon t \geq (1 - \varepsilon)|E|$$

and

$$\left| \bigcup_{Q \in W_m, Q \cap E \neq \emptyset} Q \right| \leq \left| \bigcup_{Q \in W_m, Q \subset E} Q \right| + \left| \bigcup_{Q \in A_E} Q \right| < |E| + \varepsilon t \leq (1 + \varepsilon)|E|. \quad \square$$

Lemma 8. *Let B be a translation invariant basis in \mathbb{R}^n . Then the family $\Delta(B)$ has M_{Φ_B} -property.*

Proof. Let $h > 1$ and $\varepsilon > 0$. Take $t > 1$ such that

$$\lim_{r \rightarrow 0} \frac{|\{M_B^{(tr)}(h\chi_{V_r}) > 1\}|}{|V_r|} > \frac{\Phi_B(h)}{2}.$$

Let us consider $r > 0$ for which

$$2\sqrt{n}r(1+t) < \varepsilon \quad \text{and} \quad |\{M_B^{(tr)}(h\chi_{V_r}) > 1\}| > \frac{\Phi_B(h)}{2} |V_r|. \quad (36)$$

It is easy to check that for every $f \in L(\mathbb{R}^n)$, $\delta > 0$, $x \in \mathbb{R}^n$ and $y \in \Gamma_n$,

$$M_B^{(\delta)}(f)(x) = M_{B(\gamma)}^{(\delta)}(f \circ \gamma^{-1})(\gamma(x)).$$

Therefore, we get

$$M_B^{(tr)}(h\chi_{V_r})(x) = M_{B(\gamma)}^{(tr)}(h\chi_{V_r})(\gamma(x)) \quad (x \in \mathbb{R}^n, \quad \gamma \in \Gamma_n).$$

Consequently,

$$\{M_B^{(tr)}(h\chi_{V_r}) > 1\} = \gamma\left(\{M_{B(\gamma)}^{(tr)}(h\chi_{V_r}) > 1\}\right) \quad (\gamma \in \Gamma_n). \quad (37)$$

Since the set $\{M_B^{(tr)}(h\chi_{V_r}) > 1\}$ is open, then there exists a set A that is a finite union of cubic intervals such that

$$A \subset \{M_B^{(tr)}(h\chi_{V_r}) > 1\} \quad \text{and} \quad |A| > \frac{\Phi_B(h)}{2} |V_r|. \quad (38)$$

Put $A_\gamma = \gamma(A)$ ($\gamma \in \Gamma_n$). Since sets A_γ are measurable in Jordan sense and mutually congruent, then by Lemma 7 we conclude the existence of $m \in \mathbb{N}^n$ and sets P_γ ($\gamma \in \Gamma_n$) such that

$$P_\gamma \subset A_\gamma \quad \text{and} \quad |P_\gamma| > \frac{|A_\gamma|}{2} \quad (\gamma \in \Gamma_n), \quad (39)$$

$$\{P_\gamma : \gamma \in \Gamma_n\} \subset H_m. \quad (40)$$

From (36)–(39) we have that for every $\gamma \in \Gamma_n$,

$$P_\gamma \subset \{M_{B(\gamma)}^{(\varepsilon)}(h\chi_{V_r}) > 1\}, \quad (41)$$

$$|P_\gamma| > \frac{\Phi_B(h)}{4} |V_r|, \quad (42)$$

$$V_r \cup \bigcup_{\gamma \in \Gamma_n} P_\gamma \subset V_{r(1+t)}. \quad (43)$$

Assuming $c = 1/4$, $c(h) = \frac{1}{2^n n^{n/2} (1+t)^n}$, $E = V_r$, $P_{B(\gamma)} = P_\gamma$ ($\gamma \in \Gamma_n$) and $Q = (-r(1+t), r(1+t))^n$, from (40)–(43) and (36) we conclude that the family $\Lambda = \{B(\gamma) : \gamma \in \Gamma_n\}$ has M_{Φ_B} -property. \square

From Theorem 1 on the basis of Lemma 8 we obtain the following result.

Theorem 2. *Let B be a translation invariant basis in \mathbb{R}^n . If the function Φ_B is non-regular, then for every $f \in L \setminus [\Phi_B(L)](\mathbb{G}^n)$ there exists a measure preserving and invertible mapping $\omega : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\{x : \omega(x) \neq x\} \subset \mathbb{G}^n$ and $|f| \circ \omega \in S_{\Delta(B)}$. In particular, if Φ_B additionally satisfies Δ_2 -condition at infinity, then the same conclusion is valid for every $f \in L \setminus \Phi_B(L)(\mathbb{G}^n)$.*

Corollary 1. *Let B be a translation invariant basis in \mathbb{R}^n . If the function Φ_B is non-regular, then the class $S_{\Delta(B)}$ is non-empty.*

5.2. It is true the following theorem.

Theorem 3. *Let B be a translation invariant basis in \mathbb{R}^n . If the function Φ_B is non-regular, then for every Orlicz space $\psi(L)(\mathbb{G}^n)$ with the properties: ψ satisfies Δ_2 -condition at infinity and $\lim_{h \rightarrow \infty} \frac{\psi(h)}{\Phi_B(h)} = 0$, the set $\psi(L)(\mathbb{G}^n) \setminus S_{\Delta(B)}$ is of the first category in $\psi(L)(\mathbb{G}^n)$.*

Remark 3. Theorem 3 generalizes the result of B. L6pes Melero [9] which asserts the same for the case when the following weak variant of the function Φ_B is considered

$$\tilde{\Phi}_B(h) = \overline{\lim}_{r \rightarrow 0} \frac{|\{M_B^{(hr)}(h\chi_{V_r}) > 1\}|}{|V_r|}.$$

Proof of Theorem 3. For $k \in \mathbb{N}$ by E_k denote the set of all functions $f \in \psi(L)(\mathbb{G}^n)$ for which there is a set $A = A(f) \subset \mathbb{G}^n$ and a rotation $\gamma = \gamma(f) \in \Gamma_n$ with the properties:

- 1) $|A| \geq \frac{1}{k}$;
- 2) is $x \in A$, $R \in B(\gamma)(x)$ and $\text{diam } R \leq \frac{1}{k}$, then $\frac{1}{|R|} \int_R f \leq k$.

It is easy to see that $\Psi(L) \setminus S_{\Delta(B)} = \bigcup_{k=1}^{\infty} E_k$. Therefore it suffices to prove that E_k is nowhere dense in $\psi(L)$ for every $k \in \mathbb{N}$.

Let $k \in \mathbb{N}$. First let us prove the closeness of E_k . Suppose $f_j \in E_k$ ($j \in \mathbb{N}$), $f \in \psi(L)$ and $\|f_j - f\|_{\psi(L)} \rightarrow 0$. Clearly, we can choose subsequence of $(\gamma(f_j))$ which is convergent by the natural metric in Γ_n . Without loss of generality assume that $\gamma(f_j)$ converges and let γ be its limit. By A denote the set $\overline{\lim_{j \rightarrow \infty} A(f_j)}$. Obviously, $A \subset \mathbb{G}^n$ and $|A| \geq \frac{1}{k}$. Take $x \in A$ and $R \in B(\gamma)(x)$ with $\text{diam } R < \frac{1}{k}$. Without loss of generality assume that $x \in A(f_j)$ for every $j \in \mathbb{N}$. Let us consider the set $T \in B(0)$ for which $R = x + \gamma(T)$. Put $R_j = x + \gamma(f_j)(T)$ ($j \in \mathbb{N}$). Then $R_j \in B(\gamma(f_j))(x)$ and $\text{diam } R_j \leq \frac{1}{k}$. Consequently,

$$\frac{1}{|R_j|} \int_{R_j} f \leq k.$$

Now taking into account that $|(R_j \setminus R) \cup (R \setminus R_j)| \rightarrow 0$ and $\|f_j - f\|_L \leq c\|f_j - f\|_{\psi(L)} \rightarrow 0$, we obtain

$$\frac{1}{|R|} \int_R f = \lim_{j \rightarrow \infty} \frac{1}{|R_j|} \int_{R_j} f_j \leq k.$$

Thus the closeness of E_k is proved. The next step is to prove that $\psi(L) \setminus E_k$ is dense in $\psi(L)$. Take a function $f \in \psi(L)$ and $\varepsilon > 0$. Since ψ satisfies Δ_2 -condition at infinity, then (see e.g. [10, § 4]) there is $g \in L^\infty$ with $\|f - g\|_{\psi(L)} < \varepsilon/2$. Let us consider a function $\ell \in S_{\Delta(B)}$ with $\|\ell\|_{\psi(L)} < \varepsilon/2$. Existence of such function is provided by Theorem 2. Obviously, $g + \ell \in S_{\Delta(B)}$. Now from the estimation $\|f - (g + \ell)\|_{\psi(L)} < \varepsilon$ we conclude the density of $S_{\Delta(B)}$ in $\psi(L)$. Consequently, using inclusion $S_{\Delta(B)} \subset \psi(L) \setminus E_k$ we obtain the density of $\psi(L) \setminus E_k$ in $\psi(L)$. Finally, taking into account the closeness of E_k we conclude that E_k is nowhere dense in $\psi(L)$. \square

5.3. In this section we will apply Theorems 2 and 3 for bases \mathbf{I}_n^k .

Lemma 9. *Let $\delta_1, \dots, \delta_n > 0$, $h > 1$, and let E be the set of all points $x \in \mathbb{R}^n$ such that*

$$\begin{aligned} x_1 &> \delta_1, \dots, x_n > \delta_n, \\ x_1 \cdots x_n &< h\delta_1 \cdots \delta_n. \end{aligned}$$

Then

$$\int_E \frac{1}{x_1 \cdots x_n} dx_1 \cdots dx_n > c(\ln h)^n,$$

where c is a positive number depending only on n .

Proof. For $n = 1$ the assertion is obvious. Let us consider the passing from $n - 1$ to n .

For $\delta_n < t < h\delta_n$ denote

$$A_t = \left\{ x \in \mathbb{R}^{n-1} : x_1 > \delta_1, \dots, x_{n-1} > \delta_{n-1}, x_1 \cdots x_{n-1} < \frac{h\delta_n}{t} \delta_1 \cdots \delta_{n-1} \right\}.$$

By Fubini theorem and the induction assumption we have

$$\begin{aligned} \int_E \frac{1}{x_1 \cdots x_n} dx_1 \cdots dx_n &= \\ &= \int_1^{h\delta_n} \frac{1}{x_n} \left[\int_{A_t} \frac{1}{x_1 \cdots x_{n-1}} dx_1 \cdots dx_{n-1} \right] dx_n > \\ &> \int_{\delta_n}^{h\delta_n} c_{n-1} \frac{1}{t} \left(\ln \frac{h\delta_n}{t} \right)^{n-1} dt > c_{n-1} \int_{\delta_n}^{\sqrt{h}\delta_n} \frac{1}{t} (\ln \sqrt{h})^{n-1} dt \geq c_n (\ln h)^n. \quad \square \end{aligned}$$

Lemma 10. For every k , $1 \leq k \leq n-1$, and an interval I of type

$$I = \times_{j=1}^n (a_j, a_j + \delta_j), \quad \text{where } \delta_k = \delta_{k+1} \cdots = \delta_n,$$

and $h > 1$ it is valid the estimation

$$|\{M_{\mathbf{I}_n^k}(h\chi_I) > 1\}| \geq ch(\ln h)^k |I|,$$

where c is a positive number depending only on n .

Proof. Without loss of generality assume that $h > 2^n$ and I is of type

$$I = (0, \delta_1) \times \cdots \times (0, \delta_k) \times (0, \delta_k) \times \cdots \times (0, \delta_k).$$

Let $x \in \mathbb{R}^n$ be such that

$$x_1 > \delta_1, \dots, x_k > \delta_k, \quad x_{k+1} > \delta_k, \dots, x_n > \delta_k, \quad (44)$$

$$x_1 \dots x_k [\max(x_{k+1}, \dots, x_n)]^{n-k} < h|I|. \quad (45)$$

Put

$$J = (0, x_1) \times \cdots \times (0, x_k) \times (0, \max(x_{k+1}, \dots, x_n))^{n-k}.$$

By (44) and (45), $J \supset I$ and $|J| < h|I|$. Consequently,

$$\frac{1}{|J|} \int_J h\chi_I = \frac{h|I|}{|J|} > 1.$$

Thus $M_{\mathbf{I}_n^k}(h\chi_I)(x) > 1$, and therefore

$$\{M_{\mathbf{I}_n^k}(h\chi_I)(x) > 1\} \supset \{x \in \mathbb{R}^n : x \text{ satisfies (44) and (45)}\} \equiv E.$$

Let us estimate $|E|$. Denote

$$T = \left\{ y \in \mathbb{R}^k : y_1 > \delta_1, \dots, y_k > \delta_k, \quad y_1 \cdots y_k < \frac{h}{2^k} \delta_1 \cdots \delta_k \right\},$$

$$E_y = \left\{ x \in E : (x_1, \dots, x_k) = y \right\} \quad (y \in T).$$

It is easy to see that for every $y \in T$,

$$|E_y|_{n-k} = \left(\left(\frac{R|I|}{y_1 \cdots y_k} \right)^{1/(n-k)} - \delta \right)^{n-k} > \frac{1}{2^n} \frac{R|I|}{y_1 \cdots y_k}.$$

Therefore, using Fubini theorem we have

$$\begin{aligned} |E| &> |\{x \in E : (x_1, \dots, x_k) \in T\}| = \\ &= \int_T |E_y|_{n-k} dy > \frac{h|I|}{2^n} \int_T \frac{1}{y_1 \cdots y_k} dy_1 \cdots dy_k. \end{aligned}$$

Consequently, by virtue of Lemma 10 we conclude the validity of the needed estimation. \square

Lemma 11. *For every k with $2 \leq k \leq n$ there are valid the estimations*

$$c_1 h(\ln h)^{k-1} \leq \Phi_{\mathbf{I}_n^k}(h) \leq c_2 h(\ln h)^{k-1} \quad (h > 1),$$

where c_1 and c_2 are positive numbers depending only on n .

Proof. Let $r > 0$, $h > 1$ and $Q = (-\frac{r}{\sqrt{n}}, \frac{r}{\sqrt{n}})^n$. By Lemma 10 we have

$$\begin{aligned} |\{M_{\mathbf{I}_n^k}(h\chi_{V_r}) > 1\}| &\geq |\{M_{\mathbf{I}_n^k}(h\chi_Q) > 1\}| \geq \\ &\geq ch(\ln h)^{k-1}|Q| \geq c_1 h(\ln h)^{k-1}|V_r|, \end{aligned} \quad (46)$$

where $c_1 > 0$ depends only on n . On the other hand by virtue of the well-known estimation (see [3, Chapter II, § 3])

$$|\{M_{\mathbf{I}_n^k}(f) > \lambda\}| \leq c \int_{\mathbb{R}^n} \frac{|f|}{\lambda} \left(1 + \ln \frac{|f|}{\lambda}\right)^{k-1} \quad (f \in L(\mathbb{R}^n), \lambda > 0),$$

it follows that

$$|\{M_{\mathbf{I}_n^k}(h\chi_{V_r}) > 1\}| \leq c_2 h(\ln h)^{k-1}|V_r|, \quad (47)$$

where $c_2 > 0$ depends only on n .

From (46) and (47) we conclude the validity of the lemma. \square

From Theorems 2 and 3 on the basis of Lemma 11 we obtain the following result.

Theorem 4. *Let $2 \leq k \leq n$. Then:*

- 1) *for every function $f \in L \setminus L(\ln^+ L)^{k-1}(\mathbb{G}^n)$, there exists a measure preserving and invertible mapping $\omega : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\{x : \omega(x) \neq x\} \subset \mathbb{G}^n$ and $|f| \circ \omega \in S_{\Delta(\mathbf{I}_n^k)}$;*
- 2) *for every Orlicz space $\psi(L)(\mathbb{G}^n)$ with the properties: ψ satisfies Δ_2 -condition at infinity and $\lim_{h \rightarrow \infty} \frac{\psi(h)}{h(\ln h)^{k-1}} = 0$, the set $\psi(L)(\mathbb{G}^n) \setminus S_{\Delta(\mathbf{I}_n^k)}$ is of the first category in $\psi(L)(\mathbb{G}^n)$.*

Remark 4. The first part of Theorem 1 was announced in [11].

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